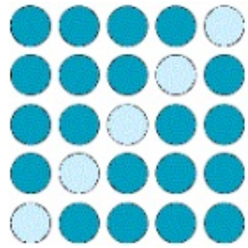
AIMSSEC

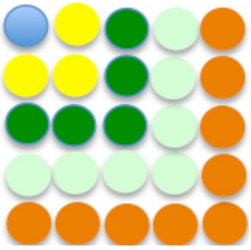
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE
TEACHER NETWORK

ODD SQUARES



(a)Think of a any whole number. Square it. Subtract the number you first thought of. Is the answer odd or even?

Try this with other numbers. What do you notice? Can you explain why? Can you find different ways to explain (and prove) the result that you noticed. You might use what you know about odd and even numbers or you might use a picture or algebra.

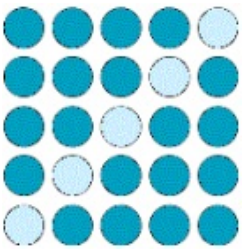


(b)Now think of a number and square it. Does this picture make you think of another number pattern? Does it work for all squares?

Solution

(a) The answer is always even.

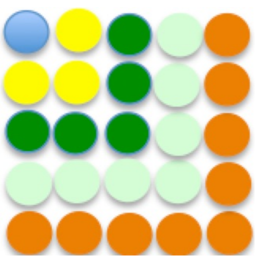
Proof 1
If the number is odd then its square is odd. The difference of two odd numbers is always even.
If the number is even then its square is even. The difference of two even numbers is always even.
So the difference between the square of any whole number and the number itself is always even.



Proof 2
A picture similar to this can be drawn for any square array. Removing the objects on a diagonal leaves two identical triangles each containing the same number of objects. So the difference between the square of any whole number and the number itself is always even.

Proof 3 Let n be any whole number
$n^2 – n = n(n-1)$
This is the product of two consecutive integers. If n is even then n-1 is odd and the other way round. The product of an odd number and an even number is even so the difference between the square of any whole number and the number itself is always even.

From the picture we also see that each triangular array is made up of 1 + 2 + 3 + … (n – 1) objects.
This shows that the sum of the series $1 + 2 + 3 + \ldots (n – 1) = ½(n^2 - n) = ½n(n – 1)$.



(b) A picture similar to this can be drawn for any square array. The discs can be arranged, as they are shown in different colours in the picture, so that there is a number of each colour for all the odd numbers 1, 3, 5, … (2n-1).

So adding up the odd numbers always gives a square number.

This shows that $1 + 2 + 3 + \ldots + (2n-1) = n^2$.

Notes for teachers

Why do this activity?

Learning Objective (Grades 8 to 10): To investigate and extend numeric and geometric patterns looking for relationships between numbers, including patterns. To represent patterns algebraically and to factorise and simplify algebraic expressions.

This activity is a wonderful example of a context in which a proof is accessible to learners via an image with algebra not being required. It would be a good choice to try with your learners once they are familiar with square numbers. As well as encouraging visualisation, it gives learners opportunities to conjecture, justify and generalise. This activity provides for learners of all abilities. All learners should have some success using words and pictures. Older and more able learners can then be encouraged to use algebra.

Possible approach

You could introduce part (a) orally. Ask each learner to think of a number and to go through the operations mentally. Invite everyone to jot down the result and share what they have with a neighbour. It might be a good idea to encourage pairs to check each other's arithmetic too! Ask pairs to talk about anything they notice about their two numbers. You could share some of these observations with the whole group. Go through the process again, asking each learner to choose a different starting number and again, to note down the end result. Do all four numbers that the pair has share any properties?

At this stage, you could collect results on the board. What do the class notice about all the numbers? Give them time to discuss in pairs or small groups why they think the result is always even. This is a chance for them to offer some suggestions, however 'polished' the explanation might be.

At this point, reveal the diagram (or draw it on the board as you go through the steps again). Without saying anything else, give the group time to discuss further. Ask each pair or small group to develop an oral explanation which they can share with everyone. As a whole class you can create an explanation together using the picture to prove that whatever the starting number, the result is always even. Learners may want to include further images.

It's important that learners understand that this will be the case *whatever the starting number*. The image given happens to be for a starting number of 5. We can't draw images for every possible starting number so how do we know the result will *always* be even? This is the key to generalisation and proof.

Now you could either ask learners to use algebra to prove the result or give learners the diagram for part (b) and ask them “Does this picture make you think of another number pattern? Does it work for all squares?”

It would be great to try and capture this for a display. You could jot down the steps of the explanations on the board as the learners build them up. Then the final versions could be put up on the wall with the problems and the images. It would be good to display any other proofs that the class has come up with.

Key questions

What do you notice about the result each time?
Will this always be the case? How do you know?
Can you describe what is happening in the image?
What is the starting number in the picture?
Can you draw similar pictures for different starting numbers?

Possible extension

Triangle Number Picture is another learning activity that focuses on visual proof. Although it leads into algebra, many learners will be able to offer written or oral proofs.

Possible support

Some learners might find it useful to use counters or cubes to represent the numbers and therefore to build up a picture of what is going on in this way. This will also allow them physically to take away the diagonal line in (a) and to build up the L shapes for the odd numbers in (b).