AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

LOOPY

$$a_{n+2} = \frac{1 + a_{n+1}}{a_n}$$

The terms of a sequence are given by the equation where each term is found from the two preceding terms.

For example if the first two terms are 2 and 5 then the next term is (1+5)/2=3 and the sequence is 2, 5, 3, 4/5, 3/5, 2, 5, ... etc.

Investigate the sequences you get when you choose your own first two terms. What happens to your sequences?

Make a conjecture about the behaviour of sequences given by this rule.

Can you prove your conjecture using algebra?

HELP

The terms a_n, a_{n+1}, a_{n+2} are three successive terms in the sequence and the subscripts n, n+1 and n+2 denote the nth term and the two terms in the sequence that follow.

So suppose $a_n = 2$ and $a_{n+1} = 5$ then $a_{n+2} = \frac{1+ a_{n+1}}{a_n} = \frac{1+5}{2} = 3$ and the next term is $\frac{1+3}{5} = \frac{4}{5}$

Now find the next 3 or 4 terms. What do you notice?

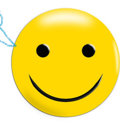
Choose your own first two terms and investigate the sequences that they start.

NEXT

The big challenge here is to use algebra to prove your conjecture. If you substitute in the formula and do the calculations correctly you will always seem to get similar patterns, but there may be exceptions.

Working with algebraic fractions follows all the same rules as working with numeric fractions.

$5^2+8^2=25+64=89$



When you have proved this conjecture you might like to try **Happy Numbers**.

Take any 2-digit number and add the squares of the digits to get the next Happy Number.

For example:

58, 89, 145, 42, 20, 4, 16, 37, 58, 89 … and now the sequence repeats in an 8-cycle

15, 26, 40, 16, 37, 58, … and this sequence starts repeating with the 4th term from 16 onwards.

23, 13, 10, 1, 1, 1, …this sequence is attracted to the fixed point 1. It just repeats 1, 1, … for ever.

What else can you discover about Happy Numbers?

NOTES FOR TEACHERS

SOLUTION

With different choices for the first two terms the 6th term is the same as the 1st and the 7th term is the same as the 2nd so we make a conjecture that this will always be so. We need algebra to investigate this.

The 1st term is a_1

The 2nd term is a_2

The 3rd term is $\frac{1+a_2}{a_1}$

The 4th term is $\frac{1+{(1+a_2)}/{a_1}}{a_2} = \frac{1+a_1+a_2}{a_1 a_2}$

The 5th term is $\frac{1+\frac{1+a_1+a_2}{a_1 a_2}}{\frac{1+a_2}{a_1}} = \frac{(1+a_1)(1+a_2)}{a_1 a_2} \times \frac{a_1}{(1+a_2)} = \frac{(1+a_1)}{a_2}$

The 6th term is $\frac{1+\frac{(1+a_1)}{a_2}}{\frac{1+a_1+a_2}{a_1 a_2}} = a_1$

The 7th term is $\frac{a_1}{\frac{(1+a_1)}{a_2}} = a_2$

So the conjecture is true **except for when zero occurs in the denominator**. So we can't have $a_1 = 0$ or $a_2 = 0$ or $a_2 = -1$, which makes the 3rd term zero, or $a_1 + a_2 = -1$, which makes the 4th term zero, $a_1 = -1$, which makes the 5th term zero.

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The decimal equivalent to the fraction $\frac{4}{33}$ is:

A. $0.433433\ldots$ or $0.\overline{433}$

B. $0.12121\ldots.$ or $0.\overline{12}$

C. $0.21212\ldots.$ or $0.\overline{21}$

D. $0.334334\ldots$ or $0.\overline{334}$

The correct answer is B

A.

$0.\overline{433} = \frac{433}{999}$

B.

$0.\overline{12} = \frac{4}{33}$

C.

$0.\overline{21} = \frac{7}{33}$

D.

$0.\overline{334} = \frac{334}{999}$

Why do this activity?
In this activity learners will get practice at substituting in formulae, calculating with fractions and making conjectures. They will find that their sequences repeat themselves in loops, or cycles, of 6 terms. This activity lends itself to mixed ability classes where most learners will enjoy working with numbers and making a discovery about cyclic patterns. The learners can be asked to explain why the patterns repeat in this way and the high flyers encouraged to use algebra to prove that the conjecture is always true except when one of the terms is zero.

Learning objectives
In doing this activity students will have an opportunity to:
- practice working with numeric and algebraic fractions;
- get experience with sequences and cyclically repeating patterns;
- to make and prove conjectures.

Generic competences (some suggestions, select from list or write your own)
In doing this activity students will have an opportunity to:
- **think mathematically,** reason logically and give explanations and proofs;
- **think flexibly**, be creative and innovative and apply knowledge and skills.

Suggestions for teaching
Your class could do the activity about recurring decimals: Divide Divide https://aiminghigh.aimssec.ac.za/years-7-9-divide-divide/ before the Loopy activity as it involves only number work and no algebra. They should investigate recurring decimals without a calculator as well as with one, and this will give them more experience of processes that involve cyclic patterns

Then start the 'Loopy' lesson with the Diagnostic Quiz as a warm-up so you can assess how well they have understood the concept of recurring decimals.

For younger learners you could use the Loopy question with your class without asking them to prove their conjecture and many learners will just work with numbers. The learners can be asked to explain why the patterns repeat in a cycle. This challenge can then be used as extension work for the high attainers and for years 11 to 13 who can be encouraged to use algebra to prove that the conjecture is always true except when one of the terms is zero.

Key questions
- What have you chosen for your first two terms? Can you use the formula to find the 3rd term?
- Now can you find the next term?
- Without doing any more calculation can you tell me the next term just looking at all the terms you have found?
- What happens if the first term is zero?
- What happens if the second term is zero?
- What happens if the first term is -1?
- What happens if the second term is -1?

Follow up
Dynamical Systems' is a whole branch of mathematics. It is a study of systems whose state evolves with time according to a fixed rule. Dynamical systems often involve repeating cycles or attraction to fixed points. This Loopy example could be the rule for some dynamical system.

See: Difference Dynamics https://nrich.maths.org/7223
Happy Numbers: https://nrich.maths.org/5408 also involves sequences that repeat themselves in cycles.
Dalmatians: https://nrich.maths.org/1926
See the articles Whole Number Dynamics I , II and III for a detailed discussion of similar problem.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6. The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is beyond the school curriculum for Grade 12 SA.
For resources for teaching A level mathematics see https://nrich.maths.org/12339

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6

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