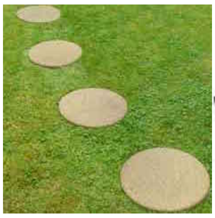
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

LEGS ELEVEN




(STEP1) Choose any two numbers from the 7 times table. Add them together. What do you notice? Repeat with some other examples, always choosing pairs of numbers from the same times table. What do you notice? Does the same thing always happen? Why or why not?

(STEP 2) Choose two digits and arrange them to make two double-digit numbers. For example if you choose 5 and 2, you can make 52 and 25. Now add your two-digit numbers. Repeat with some other examples. What do you notice? Does the same thing always happen? Why or why not?

(STEP 3) Look at this sequence of numbers: 11, 101, 1001, 10001, 100001, …
Can you describe the sequence?
Divide numbers in this sequence by 11, WITHOUT using a calculator.
What do you notice? Does the same thing always happen? Why or why not?

(STEP 4) Take any four-digit number, move the first digit to the 'back of the queue' and move the rest along. For example 5238 would become 2385.

Now add your two numbers.

Is the answer always a multiple of 11? Can you prove it?

What can you say about how this process works?

What happens when you do this with three-digit numbers? Five-digit numbers? Six-digit numbers?

Try some examples.

When does this process give a multiple of 11 and when is it not a multiple of 11?

Prove your findings.

What can you say about 38-digit numbers?

Help

To help you this puzzle has been written in short steps so that what you notice at each step should help you to do the next step. Work with a group and help each other.

Possible extension

What if you start with a three-digit number?
Or a five-digit number?
Or a six-digit number?
Or a 38-digit numbers ... ?

Can you prove your findings?

Find the numbers https://aiminghigh.aimssec.ac.za/years-6-7-find-the-numbers/

NOTES FOR TEACHERS

SOLUTION

STEP 1. Choosing any two numbers from the **same** times table, say the k times table then both numbers must be multiples of k so the sum of the numbers is also a multiple of k, that is it is also in the k times table.

STEP 2. Taking any 2-digit number, say 52, and reversing the digits we get 25. Then adding these numbers we get 52 + 25 = 77 which is a multiple of 11. This happens with every 2-digit number because we can write any 2-digit number as $10a + b$ where a and b are the digits. Reversing the digits gives the number $10b + a$ and adding the two numbers gives $(10a + b) + (10b + a) = 11(a + b)$.

STEP 3. The numbers in the sequence 11, 101, 1001, 10001,… are all of the form $10^n + 1$.
Notice that 0, 990, 99990, 9999990,… are all **divisible by 11** (that is when the number has an even number of 9's) and adding 11 to these numbers we get **SEQUENCE 3A: 11, 1001, 100001, 10000001,…**
And 90, 9990, 999990, … are **not divisible by 11** (that is when the number has an odd number of 9's) and adding 11 to these numbers we get **SEQUENCE 3B 101, 10001, 1000001, 100000001,…**
So this proves that $10^n + 1$ is divisible by 11 when n is odd and not when n is even.

STEP 4. We now investigate what happens when we take any 4-digit number and move first digit (the digit in the thousands place) to the 'back of the queue' (to the units place), move the rest along and add the two numbers. We want to know if the total is divisible by 11.
For example 5238 becomes 2385 and 5238 + 2385 = 7623 and 7623 = 11 × 1089. Remembering STEP 2 we start with the numbers $1000a + 100b + 10c + d$ and $1000b + 100c + 10d + a$.
The sum is $1001a + 1100b + 110c + 11d$ and this will always be divisible by 11, for all values of a, b, c and d, because 1001, 1100, 10 and 11 are always divisible by 11.

This process depends on place value and numbers from SEQUENCE 3A.
This process **will work for numbers with an even number of digits**, using more numbers from SEQUENCE 3A so it will work for a 38 digit number.
This process **will not work for numbers with an odd number of digits**, because the place value breakdown uses numbers from SEQUENCE 3B which are not divisible by 11.

Why do this activity?

This activity helps learners to improve their number sense and to gain an in depth understanding of place value. Challenging learners to justify and explain their findings encourages them to develop their understanding of place value and their reasoning skills at the same time. The problem is suitable for students to work on in pairs or in small groups.

Learning objectives

In doing this activity students will have an opportunity to deepen understanding of place value**.**

Generic competences

In doing this activity students will have an opportunity to:
1. think mathematically, reason logically and give explanations and proofs;
2. interpret and solve problems;
3. communicate in writing, speaking and listening; exchange ideas; criticise and present information and ideas to others; analyze, reason and record ideas effectively.

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

Which of the following numbers is divisible by 11 and also divisible by 3?

A. 909 **B. 400004** **C.1001** **D. 9009**

The correct answer is D
because **9009** $= 3 \times 3 \times 11 \times 91$

909 $= 3 \times 3 \times 101$ It is divisible by 3 but not by 11.

400004 $= 2 \times 2 \times 11 \times 9091$. It is divisible by 11 but not by 3.

1001 $= 11 \times 91$. It is divisible by 11 but not by 3.

https://diagnosticquestions.com

Suggestions for teaching

Start the lesson with the Diagnostic Quiz as a warm-up to this activity.

Then give the class a few minutes to work on STEP 1 individually. Then, as a class, invite learners to say what they have noticed and to explore the idea that when two numbers have a common factor then the sum of the numbers has the same factor. Help the learners to understand the language about divisibility – when a number N is a multiple of any number then it is divisible by that number – in other words : that number divides exactly into N without a remainder.

Some learners may wish to use a division algorithm, but you might draw their attention to the fact that they only need to confirm that there is no remainder, not find the answer to the division. One way to test for divisibility is to use 'chunking', in this case by subtracting known multiples of eleven.

For example: $7623 - 6600 = 1023$ and $1023 - 22 = 1001$ So if we find that 1001 is divisible by 11 then we know 7623 is divisible by 11.

This is STEP 2: Choose your own 4 digit number, move the first digit to the back of the queue and add the result to your original number. Check whether your answer is divisible by 11.

For the rest of the work use the **1 – 2 – 4 – more** strategy. Start the class working on STEP 2 for 5 minutes then get them working in pairs. Make sure that each pair has a copy of the whole question or write it up on the blackboard.

After another 5 – 10 minutes have a whole class discussion. This is a good moment for self checking, if anyone thinks they have a counter example then the class could do it on the board all together, and this can help to sort out any mistakes in their addition strategy or misconceptions about divisibility. Ask for explanations about why it happens that when a 2-digit number and the number formed by reversing the digits are added the sum is always divisible by 11.

Then the class should continue to work in pairs on STEP 3 and some pairs may think they have answered it and move on to STEP 4. Go around the class giving encouragement and, where you judge it to be helpful, ask key questions to help learners to make progress. Organise pairs to work with another pair to make groups of four, to compare their findings and agree the answers they want to give. Make sure that any pair who are struggling get to work with a pair who can help them.

The class should understand that your policy is, as a general rule, to pick ANY member of the class to speak for the group that they have been working with. This is a strong incentive for students who are having difficulties to persevere and try to sort out their difficulties and for those who have understood the work to explain it to the rest of their small group. This will not in any sense hold the high flyers back. On the contrary, there is no better way to gain a deep understanding of something than having to explain it to someone else and it also develops the high flyer's communication skills.

Key questions

- Will it always work? How do you know?
- How do we know if a number is divisible by 11?
- Starting with 5 thousands, 2 hundreds, 3 tens and 8 units, we could write it $5\times1000 + 2\times100 + 3\times10 + 8\times1$. After moving the digits my new number would have 2 thousands, 3 hundreds, 8 tens and 5 units, and could be written as $2 \times 1000 + 3 \times 100 + 8 \times 10 + 5 \times 1$. How many 5s, 2s, 3s and 8s would there be when we add the two numbers?"
- Can you convince the rest of your group that these are all multiples of 11 and the total must be a multiple of 11?
- Will this process always generate a multiple of 11?

Follow-up ideas

Find the numbers https://aiminghigh.aimssec.ac.za/years-6-7-find-the-numbers/
Half a Million Game https://aiminghigh.aimssec.ac.za/years-6-7-half-a-million-game/
Magic 13837 https://aiminghigh.aimssec.ac.za/years-6-10-magic-13837/
Times Nine https://aiminghigh.aimssec.ac.za/grades-6-to-9-times-nine/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **beyond** the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	Upper Primary Age 9 to 11	Lower Secondary Age 11 to 14	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6