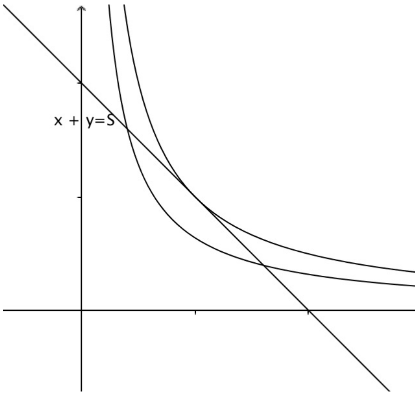
GREATEST PRODUCT

Various pairs of numbers add together to give a sum of 10:
3 + 7 = 10
2 + 8 = 10
4.4 + 5.6 = 10

The products of these numbers are all different:
3 x 7 = 21



2 x 8 = 16
4.4 x 5.6 = 24.64
What is the greatest product that can be made from pairs of numbers that add up to 10?

Which of the rectangles that have a perimeter of 20 units has the greatest area?

Try answering these questions with different starting numbers.

Can you find a strategy for splitting numbers so that you always get the greatest product?

How does this graph show the greatest product of two numbers that add up to a fixed sum?

How would you use calculus to find the greatest product of pairs of numbers that add up to any given sum?

Does this result for two numbers generalise to triples of numbers that add up to a given sum?

SOLUTION

1. By trial and improvement with different starting numbers we find that, for any number S, the greatest product of two numbers that add up to S is ½S × ½S, so for S=10 the greatest product is 25.

2. A rectangle of perimeter of 20 units and length L has breadth 10-L so, as already found, the greatest area is 25 square units when the rectangle is a square. So the largest rectangular area that can be enclosed by a fence of a given length is a square.

3. This result is shown by the graph. All the pairs of numbers that add up to the sum S are represented by points on the line x + y = S. At the points where the rectangular hyperbola xy = c cuts the line x + y = S the product x(S – x) has value c. In the graph the 'lower' hyperbola cuts the line x + y = S at two points (for example for x + y = 10 the points could be (2, 8) and (8, 2) on the hyperbola xy = 16). Note the symmetry. Symmetry is key to understanding that, to maximise c, the hyperbola giving the greatest product touches the line at (½S, ½S). For greater values of c the hyperbola does not intersect the line x + y = S

4. To maximise the product A = x(S – x) we find the point where the derivative dA/dx is zero.
dA/dx = S – 2x so dA/dx = 0 for x = ½S and this gives the maximum value of A to be $\frac{1}{4}S^2$.

5. The result does generalise for triples of numbers that add up to a fixed sum, and for sets of four or more numbers. Evidence can be found by experimenting with different numbers but the proof of this result requires more than school mathematics. For example for 3 numbers that add up to 12 the maximum product is 4×4×4 and for 4 numbers adding up to 12 the maximum product is 3×3×3×3.

6. The result also applies to the maximum volume of a cuboid shaped box where the 3 dimensions: length, breadth and height, add up to a fixed sum. The maximum volume occurs for a cube.

Proof

For a cuboid shaped box of any height the maximum volume will be when the area of the base is a maximum, that is for a square base.

So if the dimensions of the box are x, x and S-2x then the volume is given by $V = x^2(S - 2x)$.

The maximum volume occurs when the derivative dV/dx = 0 so we find the derivative $dV/dx = 2Sx - 6x^2 = 2x(S - 3x)$.

For the maximum volume $x = \frac{1}{3}S$ making all the dimensions of the cube equal.

NOTES FOR TEACHERS

Why do this activity?

The activity starts with an investigation that could be done in primary school leading to a conjecture that does in fact hold for the product of n numbers that add up to a given sum S for *any value* of n. This is particularly suitable for Grade 12 as it involves graphs, calculus and optimisation and shows connections between number, algebra, geometry, calculus and optimisation. It is important to draw learners' attention to the many connections between the different mathematical ideas. The activity also requires learners to interpret the information given and write it in the form of algebraic expressions (quadratic or cubic) which they can differentiate. This activity introduces learners to some of the methods of optimisation which is an important topic in mathematics.

Intended Learning Objectives (Grades 12)

To use differentiation to determine the coordinates of stationary points. To solve problems concerning optimization. To appreciate the connections between the numerical, algebraic and geometrical aspects of the same result.

Possible approach

The parts of this question are numbered 1 to 6 in the solution and you can plan which parts to include in your lesson according to the learning objectives you want to achieve and the time available. Other parts can be used as extension activities for high flyers or given as homework.

Part 1 could be a lesson starter and you might write this on the board and ask learners to work on it individually and then have the learners share their ideas and discuss the results until they agree on a conjecture.

You could give the other parts of the activity to different groups of learners and then ask learners to present and explain their work to the whole class.

Key questions

The rectangle

If you know the length and perimeter what is the breadth? What is the area?

The graphs of $x + y = S$ and $xy=c$

What can you say about the intersections of the hyperbola with the line?

How do you know that the hyperbola touching the line gives the greatest product?

Calculus proof
If one of the two numbers is x and the sum is S what is the other number?
Can you write an expression for the product A as a function of x?
What is the gradient of the graph of A(x) when the derivative dA/dx=0?
Can you check whether this gives a maximum or a minimum?

The box with maximum volume
How will you make the base of the box have a maximum area?
If length + breadth + height = S and the base has a maximum area what are the 3 dimensions?
Can you write the volume V as a cubic in one variable x?
What is the gradient of the graph of V(x) when the derivative dV/dx=0?
Can you check whether this gives a maximum or a minimum?

Possible extension

Consider sets of positive real numbers $a_1, \ldots, a_n$, for any positive integer *n*.

The arithmetic mean of the given numbers is defined as

$A(a) = (a_1 + , \ldots, + a_n)/\mathrm{n}$

and their geometric mean is given by

$G(a) = (a_1 \times , \ldots, \times a_n)^{1/\mathrm{n}}$

The arithmetic mean is always greater than or equal to the geometric mean, that is $A(a) \geq G(a)$.

or $(a_1 + , \ldots, + a_n)/\mathrm{n} \geq (a_1 \times , \ldots, \times a_n)^{1/\mathrm{n}}$

This result is called the *Arithmetic Mean - Geometric Mean Inequality* (*AM-GM* for short).
How is the AM-GM inequality equivalent to saying that the maximum product of numbers that add up to a fixed sum occurs when all the numbers are equal?

Possible support

Learners could start with using calculus to maximise the area of a rectangle with a given perimeter. This leads naturally to writing an expression for the area.

Then they could tackle the number problem (1) and if they don't see how to prove the conjecture ask them if they have seen anything like it before.