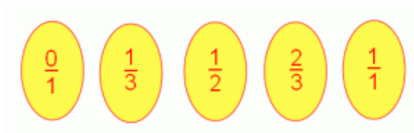
AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

FAREY SEQUENCES

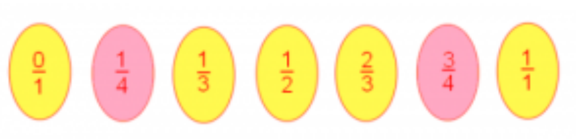
It is not difficult to put a list of decimals in order of size. But what about ordering fractions?

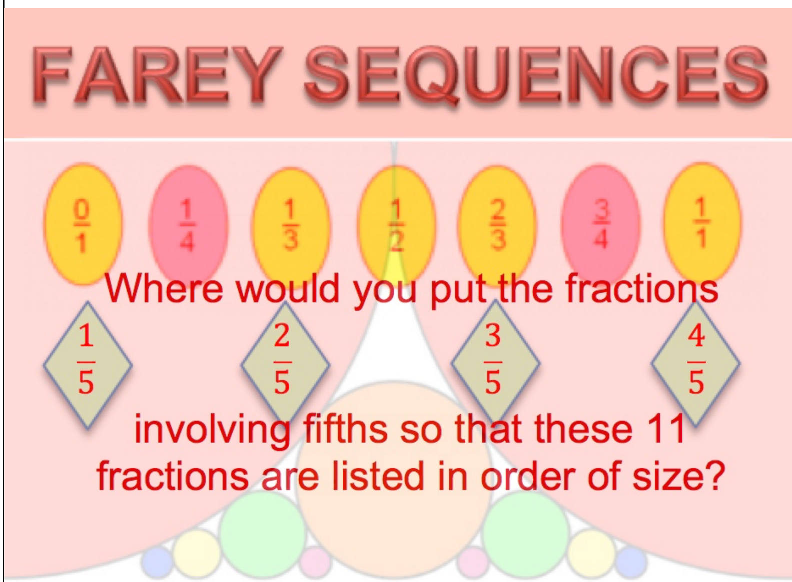
In 1816 John Farey introduced a method for producing sequences of fractions in order of size. Can you discover his method from the following examples?



The third Farey sequence shown here lists in order, in their simplest form, all the fractions between 0 and 1 that have denominators 1, 2 and 3.

The fourth Farey sequence (F4) shown here lists in order, in their simplest form, all the fractions between 0 and 1 that have denominators 1, 2, 3 and 4.

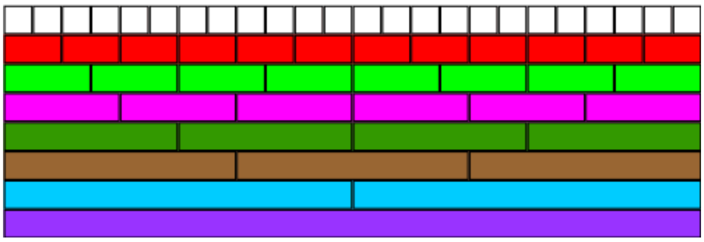




Can you find the fifth Farey sequence?

What about more sequences in this sequence of sequences?

HELP



A fraction wall might help you to match up equivalent fractions.

Notice that in the 4th Farey sequence, F4, $\frac{3}{4}$ is slotted in between $\frac{2}{3}$ and $\frac{1}{1}$

What do you notice about the fractions on either side when you slot in a new fraction?

Choose any three consecutive fractions from a Farey Sequence.

Can you find a way to combine the two outer fractions to make the middle one?

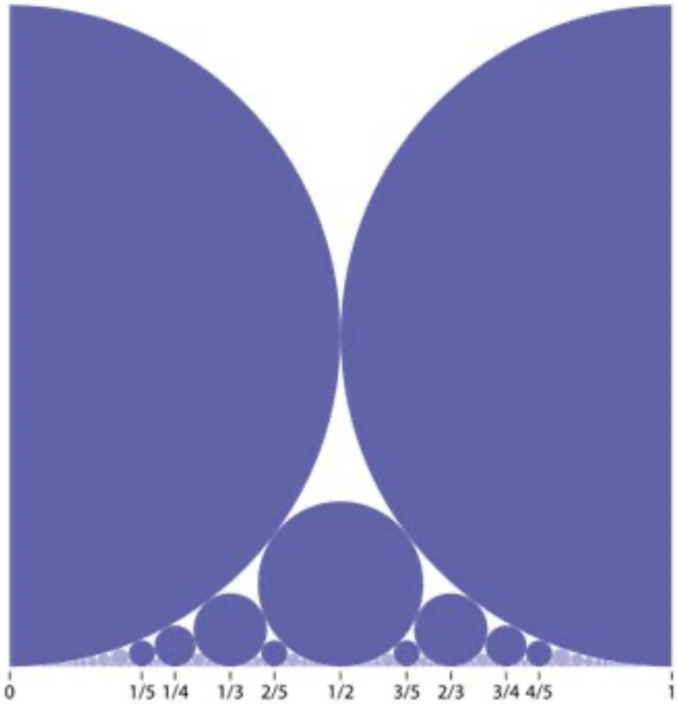
NEXT

Investigate more sequences.

Can you find a Farey sequence with an even number of fractions? Is there more than one?

Why or why not?

Can you find some examples of sequences that have a lot of extra fractions that were not in the sequence before, and others that only have a few extra fractions. Can you explain why this happens?



Mathematics is full of amazing patterns and there is a lot more mathematics involving Farey sequences. What do you notice in this picture?

NOTES FOR TEACHERS

SOLUTION

The fifth Farey sequence is $\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{1}{1}$

The sixth Farey sequence is $\frac{0}{1},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{1}{1}$

The seventh Farey sequence is $\frac{0}{1},\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{2}{7},\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{3}{5},\frac{2}{3},\frac{5}{7},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{6}{7},\frac{1}{1}$

Here are some extension questions and some solutions:

(1) There are lots of extra fractions in F11that are not in F10.

The extra fractions are $\frac{1}{11},\frac{2}{11},\frac{3}{11},\frac{4}{11},\frac{5}{11},\frac{6}{11},\frac{7}{11},\frac{8}{11},\frac{9}{11},\frac{10}{11}$

(2) There are only a few extra fractions in F12 that are not in F11.

The extra fractions are $\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$

(3) Can you explain why this is the case?

This is because 11 is a prime number and so none of the fractions with 11 in the denominator will have already occurred and been cancelled to the equivalent fraction with a smaller denominator already in the sequence. In contrast all the twelfths $\frac{2}{12},\frac{3}{12},\frac{4}{12},\frac{6}{12},\frac{8}{12},\frac{9}{12},\frac{10}{12}$ with numerators having common factors to 12 will already have occurred in a previous sequence.

(4) When will you need lots of extra fractions to get the next Farey sequence?
The seventeenth Farey sequence will need many extra fractions as 17 is a prime number.

(5) Will every Farey Sequence be longer than the one before? How do you know?
Yes every Farey sequence will be longer than the one before. The nth Farey sequence will contain $\frac{1}{n}$ whereas all the earlier Farey sequences contain fractions with denominators less than n.

(6) So far, all the Farey Sequences have contained an odd number of fractions. Can you find a Farey Sequence with an even number of fractions?

The only Farey sequence with an even number of fractions is the first $\frac{0}{1},\frac{1}{1}$

(7) Can you find more than one example where you put in an odd number of fractions to get the next Farey Sequence? If not, why not?

This is quite a challenging question. Putting in ½ between 0/1 and 1/1 to get the second Farey sequence is the only time you put in an odd number of fractions to get the next one in the sequence. The extra fractions added to make the nth Farey sequence come in pairs a/n and (n-a)/n

(8) What do you notice about the fractions on either side when you slot in a new fraction to make the next sequence?

For example, in F4, $\frac{3}{4}$ *slotted in between* $\frac{2}{3}$ *and* $\frac{1}{1}$

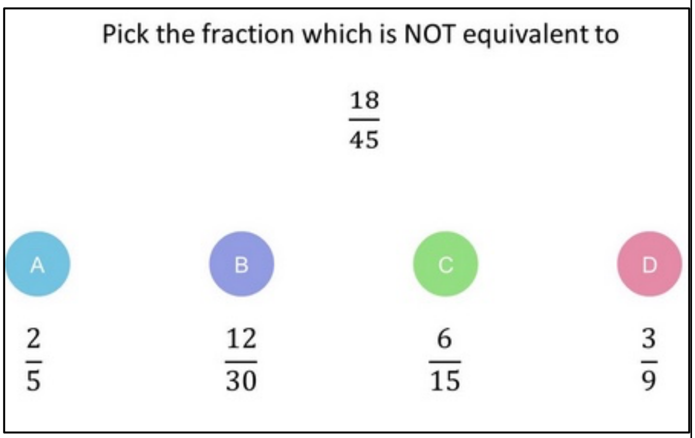
Choose any three consecutive fractions from a Farey Sequence. Can you find a way to combine the two outer fractions to make the middle one?
To get the numerator and denominator of each fraction you add the numerators and denominators of the fractions on either side, for example to get $\frac{3}{4}$ in F4 you add the numerators and denominators of $\frac{2}{3}$ and $\frac{1}{1}$: $\frac{2+1}{3+1} = \frac{3}{4}$ **so this rule has its rightful place in mathematics although it is sometimes misused.**

It certainly does NOT give the sum of two fractions which is bigger than the fractions and not somewhere between them.

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.
5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.



The correct answer is D because 3/9 = 15/45 not 18/45
All the other fractions given are equivalent to 18/45 = 2/5 in the lowest terms.
https://diagnosticquestions.com

Why do this activity?

This activity involves a pleasing pattern and many learners will enjoy investigating and making conjectures about this intriguing sequence of sequences. This activity offers the opportunity to practise the important skill of ordering fractions. Although Farey sequences are not on the examination syllabus, learners do need to have a good understanding of fractions, and this type of investigation helps learners to deepen their understanding of the concept.

Teachers can plan lessons to suit learners of different attainment levels as this activity provides a chance to work on a variety of questions at different levels (see the additional questions and solutions above).

Learning objectives

In doing this activity students will have an opportunity to deepen their understanding of equivalent fractions and about comparing the sizes of fractions.

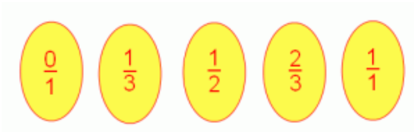
Generic competences

In doing this activity students will have an opportunity to **think flexibly**, be creative and innovative and apply knowledge and skills.

Suggestions for Teaching

Display the third Farey sequence on the board and ask learners:

"This is the third Farey sequence. Can you work out what rules have been used to generate it?"

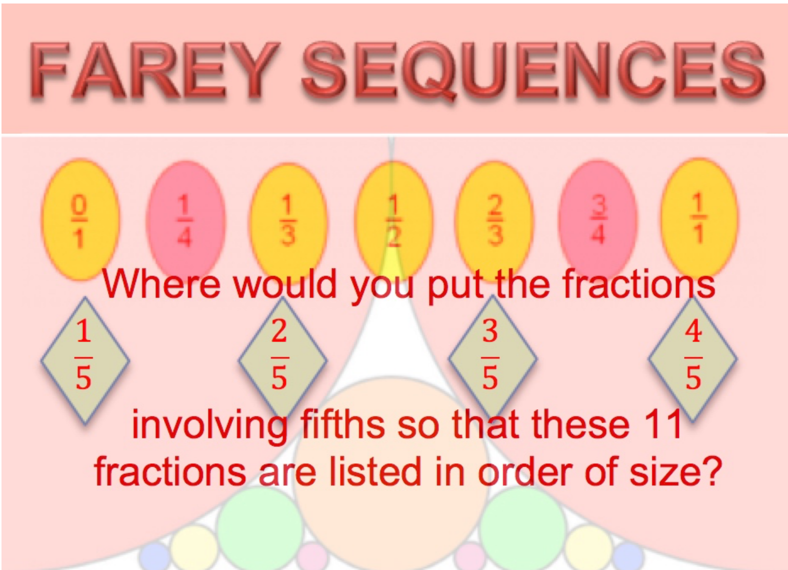


Once students have identified all the criteria, ask them to discuss with their partner what they think the fourth Farey Sequence will look like. Then show them the fourth sequence:

Perhaps clarify the rule about equivalent fractions by asking

"Where is $\frac{2}{4}$?"

If possible show this picture or give out the worksheet (page 1).



Write up $F2$, $F3$, and $F4$ on the board, and challenge learners to work out $F5$, $F6$, and $F7$, using the agreed rules, and think about what they will do next:

"When you've finished, I'll be asking you to investigate these sequences, so think about questions you would like to ask and things that you notice while you are working."

As learners are working on the sequences, circulate to see if everyone is getting the same results. If so, when the class is ready to move on write the agreed results for $F5$, $F6$, and $F7$ on the board. If not, ask learners with different answers to write their sequences on the board, and ask the class for their comments. When consensus is reached, move on:

"Mathematicians often look for patterns to help them to understand something better. What might mathematicians notice about the Farey sequences we have found? What questions might they want to explore next?"

Take suggestions from the class and list them on the board. There are some questions to consider, and solutions, above which could be used to supplement the class's suggestions.

Allow the learners to choose what they would like to explore. They may wish to work with a partner. One nice way to feed back at the end of this activity is for each learner to work on paper and for findings on similar conjectures to be displayed together on a noticeboard or on the classroom wall.

Key questions

How do you compare the sizes of two fractions?

What are those two fractions written as fractions with a common denominator?

What are those three fractions written as fractions with a common denominator?

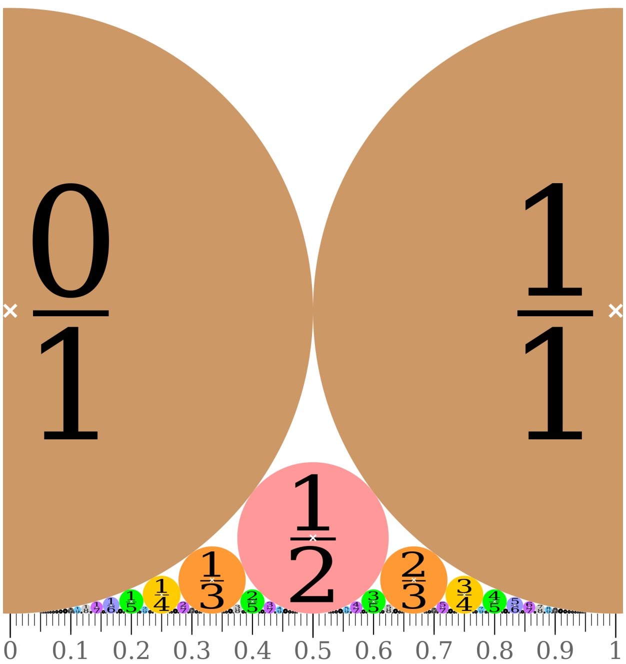
How are you going to find out if that fraction goes between those two other fractions?

When is $\frac{a}{b} < \frac{c}{d}$? (answer when $\frac{ad}{bd} < \frac{bc}{bd}$ which means that $ad<bc$)

Follow up

You may like to tell students that there is a lot more mathematics involving Farey sequences. Proving the results goes beyond school mathematics but that should not stop us enjoying a picture.

You might show them this picture. The line at the bottom is a tangent to all the circles at the points occurring in a Farey sequence and all the circles in the picture just touch the other circles around them.



If you magnify the picture more and more, smaller and smaller circles will be revealed, all tangent to the line and tangent to each other. These are called Ford circles. There are infinitely many Ford circles **one corresponding to every rational number**.

This links to hyperbolic geometry and a lot more mathematics.

https://en.wikipedia.org/wiki/Ford_circle

Mediant Madness https://nrich.maths.org/831

Ford Circles https://nrich.maths.org/6594

Farey Neighbours https://nrich.maths.org/6598

Go to the **AIMSSEC AIMING HIGH** website for lesson ideas, solutions and curriculum links: http://aiminghigh.aimssec.ac.za

Subscribe to the **MATHS TOYS YouTube Channel** https://www.youtube.com/c/MathsToys/videos

Download the whole AIMSSEC collection of resources to use offline with the **AIMSSEC App** see https://aimssec.app or find it on Google Play.

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and school years up to Secondary 5 in East Africa.
New material will be added for Secondary 6.
For resources for teaching A level mathematics (Years 12 and 13) see https://nrich.maths.org/12339
Mathematics taught in Year 13 (UK) & Secondary 6 (East Africa) is beyond the SA CAPS curriculum for Grade 12

	Lower Primary Approx. Age 5 to 8	Upper Primary Age 8 to 11	Lower Secondary Age 11 to 15	Upper Secondary Age 15+
South Africa	Grades R and 1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
East Africa	Nursery and Primary 1 to 3	Primary 4 to 6	Secondary 1 to 3	Secondary 4 to 6
USA	Kindergarten and G1 to 3	Grades 4 to 6	Grades 7 to 9	Grades 10 to 12
UK	Reception and Years 1 to 3	Years 4 to 6	Years 7 to 9	Years 10 to 13