AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
SCHOOLS ENRICHMENT CENTRE (AIMSSEC)
AIMING HIGH

BALANCE POWER

You can weigh all integer masses from 1 to 60 with 6 weights putting the weights in one pan and the object in the other pan. Which weights are used?

What about weighing all integer masses from 1 to 1000?

What about from 1 to n?

What could this have to do with the way computers work?

For a bigger challenge can you find how many weights are needed if you put weights in both pans?

Help

Explain why this list shows how to weigh masses 1, 2, 3, 4, 5, 6, and 7 units and ask learners to continue the list for masses 8, 9, 10 etc.

Weights
1
1 and 2
1 + 2 = **3**
1, 2 and 4
1 + 4 = **5**
2 + 4 = **6**
1 + 2 + 4 = **7**
1, 2, 4 and 8?

Extension

We use a decimal number system with units, tens, hundreds, thousands … etc (all powers of 10). But 10 is **not** the only number that can be used as the base for a number system. Other numbers can be used e.g. 2 or 3 or any other number. The ancient Babylonians had a base 60 number system and we still have traces of it with seconds and minutes. Computers use a base 2 number system.

The bigger challenge is to see how many weight you need to balance every each 1, 2, 3, … , 13 units using both pans. What pattern do you notice in the weights used? Can you explain it?

The weights can be coded:
- either as +1 times the weight when the weight is in the **opposite** pan to the object,
- or 0 times the weight when the weight is **not used** at all,
- or -1 times the weight when the weight is in the **same** pan as the object.

We are using a numbering system

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

where the coefficients $a_0, a_1, a_2, a_3,$ … are all 1, 0 or -1 and x=3.

This is equivalent to the base 3 number system.

All integers can be written in base 3 with digits 0, 1 and 2, that is as

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$$

where the coefficients $b_0, b_1, b_2, b_3,$ ... are all 0, 1 or 2 and x=3. This explains why all objects weighing an integer number of units can be weighed with a set of weights that are all powers of 3.

NOTES FOR TEACHERS

SOLUTION

When a problem seems difficult it is often a good idea to start with a simple case. Here think about what masses you can find with weights of 1 and 2. With these weights you can weigh masses of 1, 2 and 3 units but no more. Then you don't need a weight of 3 units.

With weights of 1, 2 and 4 units masses 1, 2, 3, 4, 1+4=5, 2+4=6 and 1+2+4=7 can be weighed so you don't need weights of 3, 5, 6 and 7 units.

With weights of 1, 2, 4 and 8 units masses 1, 2, 3, 4, 5, 6, 7, 1+8=9, 2+8=10, 1+2+8=11, 4+8=12, 1+4+8=13, 2+4+8=14, and 1+2+4+8=15 can be weighed so you don't need weights of 9, 10, 11, 12, 13, 14, or 15 units.

There seems to be a pattern – the only weights needed are 1, 2, 4, 8, 16, … that is the powers of 2, and no other weights are needed.

So with 1, 2, 4, 8 and 16 all integer masses up to 31 units can be weighed.

With 1, 2, 4, 8, 16 and 32 all masses up to 63 units can be weighed.

Binary arithmetic explains this – all whole numbers can be expressed in binary arithmetic in terms of powers of 2 as: 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1011, …

From right (the units digit) to left the columns represent 2^0, 2^1. 2^2, 2^3, … and the binary number is written as a sum of powers of 2 using only the digits 0 and 1.

Masses up to $2^n - 1$ can be weighed with n weights. So for a mass of 1000 units we need 9 weights (1, 2, 4, 8, 16, 32, 64, 128, 256 and 512) and with those weights masses of up to 1023 units can be weighed.

60 = 32 + 16 + 8 + 4 and 1000 = 512 + 256 + 128 + 64 + 32 + 8

If it is possible to put weights on both sides, you only need to use powers of 3. With weights of 1 and 3 you can weigh masses 1, 2, 3 and 4 units and you don't need a weight of 2 units because you can get 2 by putting 3 on one side and 1 on the other (3-1).

The next weight needed is 9 because you can put 1 and 3 and one side and 9 on the other to get 5.

With weights 1, 3 and 9 you can weigh up to 13 and the next weight needed is 27 because 27-13 = 14.

To weigh integers masses between 1 and 60, you only need 5 weights, 1, 3, 9, 27, 81. These weights will weigh up to 242 units. This method uses base 3 numbers.

60 = 81 + 9 – 27 – 3 and 1000 = 729 + 243 + 27 + 1

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.

Binary Code

If 10 represents the number 2 in binary code
and 100 represents the number 4
and 101 represents the number 5
what does 111 represent?

A. 6 **B.** 3 **C.** 7 **D.** 8

5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The correct answer is **C**
because $7 = 4 + 2 + 1 = 2^2 + 2^1 + 2^0$ in binary code (or base 2) .
Possible misconceptions:
A and **D** Students giving these answers are probably just guessing.
B. Maybe these students added 1 + 1 + 1 = 3

https://diagnosticquestions.com

Why do this activity?

The problem can be used to reinforce understanding of place value. Learners can discover the base two number system for themselves and gain experience of working with powers of 2 that can be related to the use of the binary system in digital computing. If you want to extend this activity the generalisation to the problem which allows weights in both pans leads naturally to representing numbers in base 3.

Learning objectives

In doing this activity students will have an opportunity to:

- reinforce understanding of place value;
- appreciate that binary numbers are useful in computing.

Generic competences

In doing this activity students will have an opportunity to:

- think critically/mathematically;
- apply knowledge and skills;
- solve and interpret problems in a variety of situations.

Suggestions for teaching

As a class, work together through weighing 1, 2, 3, 4, 5 and 6 units with the fewest weights possible and then ask the learners to continue the process.

Discuss with the class how the 2-state system, on or off, enables all numbers to be recorded and how this is the basis of the working of computers.

Key questions

Weights in one pan only

What weights are needed to weigh 1, 2, 3,....15 units?
What pattern do you notice in the weights used? Can you explain it?
When you include another weight in the set of weights used how many objects can now be weighed?
How many weights are needed to weigh objects of 1, 2, 3 ... 100 units?
How many weights are needed to weigh 1 to 1000 units or 1 to n units?

Weights in both pans

What weights are needed to weigh 1, 2, 3,....13 units?
What pattern do you notice in the weights used? Can you explain it?
When you include another weight in the set of weights used how many objects can now be weighed?
How many weights are needed to weigh objects of 1, 2, 3 ... 100 units?
How many weights are needed to weigh 1 to 1000 units or 1 to n units?

Follow up

Legs Eleven https://aiminghigh.aimssec.ac.za/years-10-12-legs-eleven/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **beyond** the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	**Upper Primary Age 9 to 11**	**Lower Secondary Age 11 to 14**	**Upper Secondary Age 15+**
South Africa	**Grades R and 1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
USA	**Kindergarten and G1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
UK	**Reception and Years 1 to 3**	**Years 4 to 6**	**Years 7 to 9**	**Years 10 to 13**
East Africa	**Nursery and Primary 1 to 3**	**Primary 4 to 6**	**Secondary 1 to 3**	**Secondary 4 to 6**