ALWAYS PERFECT

$2 \times 3 \times 4 \times 5 + 1 = 11 \times 11$

$21 \times 22 \times 23 \times 24 + 1 = 505 \times 505$

Pick your own four consecutive numbers, find their product and add one.
Is your answer a perfect square?

Show that if you add 1 to the product of four consecutive numbers the answer is ALWAYS a perfect square

Help

It will help you to learn how to factorize the difference of two squares.

$(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$

so we call $a^2 - b^2$ 'the difference of two squares' and factors are given by $a^2 - b^2 = (a + b)(a - b)$

See: Difference of Squares: https://aiminghigh.aimssec.ac.za/grades-8-to-10-differences-of-squares/

Extension

Work out the sequence of answers you get when you find the square roots of
$1 \times 2 \times 3 \times 4 + 1$
$2 \times 3 \times 4 \times 5 + 1$
$3 \times 4 \times 5 \times 6 + 1$ … etc.
What do you notice about this sequence? Can you find a formula for the nth term?
[The sequence of results is given by the formula 2n + 4 for the nth term.]

Also see: Take Three from Five https://aiminghigh.aimssec.ac.za/grades-8-to-12-take-three-from-five/

NOTES FOR TEACHERS

SOLUTION

Method 1

In order to make the algebra simple we take the 4 consecutive numbers as $(x - 1)$, x, $(x + 1)$ and $(x + 2)$.

Multiplying the 4 numbers and adding 1 we get:

$(x - 1)\, x\, (x + 1)(x + 2) + 1 = x\,(x^2 - 1)\,(x + 2) + 1$

$= x^4 + 2x^3 - x^2 - 2x + 1$

Factorising this gives $(x^2 + x - 1)(x^2 + x - 1)$ which is a perfect square for all values of x.

Alternatively $x(x + 1)(x + 2)(x + 3) + 1 = x^4 + 6x^3 + 11x^2 + 6x + 1 = (x^2 + 3x + 1)^2$

Method 2

Notice that $2 \times 5 = 10$ and $3 \times 4 = 12$ and $2 \text{ x } 3 \times 4 \times 5 = 11 \times 11$ and

$21 \times 24 = 504$ and $22 \times 23 = 506$ and $21 \times 22 \times 23 \times 24 = 505 \times 505$.

We can make the conjecture that

$x(x + 1)(x + 2)(x + 3) + 1$

$[x(x + 3) + 1]^2$

$[(x + 1)(x + 2) - 1]^2$

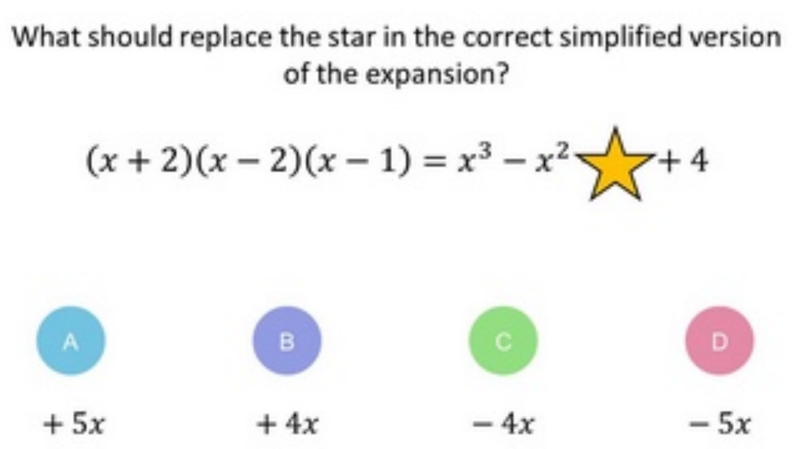
are all equal. This can easily be proved by multiplying out the three expressions.

This shows that the product of four consecutive integers plus one is always a perfect square and it gives the factors of the quartic without requiring factorisation.

Diagnostic Assessment

This should take about 5–10 minutes.

1. Write the question on the board, say to the class:
   "Put up 1 finger if you think the answer is A, 2 fingers for B, 3 fingers for C and 4 fingers for D".
2. Notice how the learners responded. Ask a learner who gave answer A to explain why he or she gave that answer and DO NOT say whether it is right or wrong but simply thank the learner for giving the answer.
3. Then do the same for answers B, C and D. Try to make sure that learners listen to these reasons and try to decide if their own answer was right or wrong.
4. **Ask the class again to vote for the right answer by putting up 1, 2, 3 or 4 fingers. Notice if there is a change and who gave right and wrong answers.** It is important for learners to explain the reason for their answer otherwise many learners will just make a guess.



5. If the concept is needed for the lesson to follow, explain the right answer or give a remedial task.

The correct answer is C

$\mathbf{(x + 2)(x - 2)(x - 1) = (x^2 - 4)(x - 1)}$

A, B and **D** The student probably got one of the signs wrong in doing the multiplication.

https://diagnosticquestions.com

Why do this activity?

This activity provides an opportunity for learners to experiment with numbers and then to use algebra to prove that a result works in general. It requires learners to factorize a quartic expression into two quadratics but looking for identical quadratic expressions makes the quartic easier to factorize.

Learning objectives

In doing this activity students will have an opportunity to:
- observe number patterns and to make and prove conjectures based on those patterns;
- practice multiplying algebraic expressions and factorizing polynomials.

Generic competences

In doing this activity students will have an opportunity to:
- think critically/mathematically;
- reason logically – to be creative and innovative - to apply knowledge and skills.

Suggestions for teaching

Ask learners to choose four consecutive numbers less than 12, to find their product and add 1. Is it a perfect square? Write some examples on the board. Ask them to do other examples with bigger numbers using a calculator.

Ask the learners what they notice about the answers. When the class believe that the answers will always be a perfect square ask them to work in pairs to prove the conjecture that the result is always a perfect square whatever numbers are chosen

If some learners find it difficult to get started you could suggest that they might find a shortcut by playing with the numbers in pairs.

Some learners will try to use algebra by multiplying x, $(x + 1)$, $(x + 2)$ and $(x + 3)$. If they don't make any progress you might suggest that the 4 consecutive numbers could be given by $(x – 1)$, x, $(x + 1)$ and $(x + 2)$.

If some learners struggle to find the factors of the quartic remind them that they are looking for two identical factors.

When some of the learners have found the factors ask them to write all their working on the board and stop and explain each step. Go over each step repeating the explanation, asking other learners if they can explain the step in different ways. Help everyone to understand and give time for copying the working into their notebooks.

If time allows all the methods can be discussed.

Key questions

- You have to show that this quartic is a perfect square. How will you do that?
- What do you know about the quadratic factors of the quartic?
- What possible constant terms can you have in the factors?

Follow up

Take Three from Five https://aiminghigh.aimssec.ac.za/grades-8-to-12-take-three-from-five/

Note: The Grades or School Years specified on the AIMING HIGH Website correspond to Grades 4 to 12 in South Africa and the USA, to Years 4 to 12 in the UK and up to Secondary 5 in East Africa. New material will be added for Secondary 6.
For resources for teaching A level mathematics see https://nrich.maths.org/12339
Note: The mathematics taught in Year 13 (UK) and Secondary 6 (East Africa) is **beyond** the school curriculum for Grade 12 SA.

	Lower Primary or Foundation Phase Age 5 to 9	**Upper Primary Age 9 to 11**	**Lower Secondary Age 11 to 14**	**Upper Secondary Age 15+**
South Africa	**Grades R and 1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
USA	**Kindergarten and G1 to 3**	**Grades 4 to 6**	**Grades 7 to 9**	**Grades 10 to 12**
UK	**Reception and Years 1 to 3**	**Years 4 to 6**	**Years 7 to 9**	**Years 10 to 13**
East Africa	**Nursery and Primary 1 to 3**	**Primary 4 to 6**	**Secondary 1 to 3**	**Secondary 4 to 6**